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On eigenvalue intervals and twin eigenfunctions of higher-order boundary value problems

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Abstract

In this paper we shall consider the boundary value problem

$$y^{(n)}+\lambda Q(t,y,y',\ldots,y^{(n-2)})=\lambda P(t,y,y',\ldots,y^{(n-2)}),\quad n\geqslant 2,\ t\in(0,1),$$

$$y^{(i)}(0)=0,\quad 0\leqslant i\leqslant n-3,$$

$$\alpha y^{(n-2)}(0)-\beta y^{(n-1)}(0)=0,$$

$$\gamma y^{(n-2)}(1)+\delta y^{(n-1)}(1)=0,$$

where $\lambda>0$, α,β,γ and δ are constants satisfying $\alpha\gamma+\alpha\delta+\beta\gamma>0$, $\beta,\delta\geqslant 0$, $\beta+\alpha>0$ and $\delta+\gamma>0$. Intervals of λ are determined to ensure the existence of a positive solution of the boundary value problem. For $\lambda=1$, we shall also offer criteria for the existence of two positive solutions of the boundary value problem. In addition, upper and lower bounds for these positive solutions are obtained for special cases. Several examples are included to dwell upon the importance of the results obtained. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

In this paper we shall consider the nth-order differential equation

$$y^{(n)}+\lambda Q(t,y,y',\ldots,y^{(n-2)})=\lambda P(t,y,y',\ldots,y^{(n-2)}),\quad t\in(0,1), \tag{1.1}$$

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together with the boundary conditions

$$y^{(i)}(0)=0, \quad 0 \leqslant i \leqslant n-3, \tag{1.2}$$

$$\alpha y^{(n-2)}(0)-\beta y^{(n-1)}(0)=0, \tag{1.3}$$

$$\gamma y^{(n-2)}(1)+\delta y^{(n-1)}(1)=0, \tag{1.4}$$

where $n \geqslant 2$, $\lambda > 0$, α, β, γ and δ are constants so that

$$\rho=\alpha\gamma+\alpha\delta+\beta\gamma>0 \tag{1.5}$$

and

$$\beta \geqslant 0, \quad \delta \geqslant 0, \quad \beta+\alpha>0, \quad \delta+\gamma>0. \tag{1.6}$$

It is noted that condition (1.6) allows α and γ to be negative.

Throughout, we shall assume that there exist continuous functions $f:(0,\infty)\to(0,\infty)$ and $p, p_1, q, q_1:(0,1)\to\mathbb{R}$ such that
(A1) for $u \in (0,\infty)$,

$$q(t) \leqslant \frac{Q(t,u,u_1,\ldots,u_{n-2})}{f(u)} \leqslant q_1(t), \qquad p(t) \leqslant \frac{P(t,u,u_1,\ldots,u_{n-2})}{f(u)} \leqslant p_1(t);$$

(A2) $q(t)-p_1(t)$ is nonnegative and is not identically zero on any subinterval of $(0,1)$.

Further, we denote

$$f_0=\lim_{u\to 0^+}\frac{f(u)}{u} \quad \text{and} \quad f_\infty=\lim_{u\to\infty}\frac{f(u)}{u}.$$

By a *positive solution* y of (1.1)–(1.4), we mean a nontrivial $y \in C^{(n)}(0,1)\cap C^{(n-1)}[0,1]$ satisfying (1.1)–(1.4), y is nonnegative on $[0,1]$ and is positive on some subinterval of $[0,1]$. If, for a particular λ the boundary value problem (1.1)–(1.4) has a positive solution y, then λ is called an *eigenvalue* and y a corresponding *eigenfunction* of (1.1)–(1.4). We let E be the set of eigenvalues of (1.1)–(1.4), i.e.,

$$E=\{\lambda>0 \mid (1.1)\text{–}(1.4) \text{ has a positive solution}\}.$$

The first contribution in this paper is the establishment of explicit intervals of λ so that the boundary value problem (1.1)–(1.4) has a positive solution. Next, for $\lambda=1$ we shall investigate the existence of twin eigenfunctions of (1.1)–(1.4). Our third contribution is the derivation of upper and lower bounds for the two eigenfunctions for special cases of (1.1)–(1.4). Specifically, we shall consider the following differential equations:

$$y''+h(t)(y^a+y^b)=0, \quad t\in(0,1) \tag{1.7}$$

and

$$y''+h(t)e^{\sigma y}=0, \quad t\in(0,1) \tag{1.8}$$

subject to the boundary conditions (1.3), (1.4) (when $n=2$). In (1.7) and (1.8), it is assumed that $0 \leqslant a < 1 < b$, $\sigma > 0$, and $h(t)$ is nonnegative on $[0,1]$ and is positive on $(0,1)$. We remark that the importance of (1.7), (1.3), (1.4) and of the discrete version of its particular cases have been well illustrated in [25, 7], respectively. With $h(t)$ being a constant function and $\alpha=\gamma=1, \beta=\delta=0$, the boundary value problem (1.8), (1.3), (1.4) actually arises in applications involving the diffusion of heat generated by positive temperature-dependent sources [1]. For instance, if $\sigma=1$ the boundary value problem occurs in the analysis of Joule losses in electrically conducting solids as well as in frictional heating.

The motivation for the present work stems from many recent investigations. In fact, when $n=2$ the boundary value problem (1.1)–(1.4) models a wide spectrum of nonlinear phenomena such as gas diffusion through porous media, nonlinear diffusion generated by nonlinear sources, thermal self-ignition of a chemically active mixture of gases in a vessel, catalysis theory, chemically reacting systems, adiabatic tubular reactor processes, as well as concentration in chemical or biological problems, where only positive solutions are meaningful, e.g., see [6, 10, 12, 13, 21, 24, 34]. For the special case $\lambda=1$, (1.1)–(1.4) and its particular and related cases have been the subject matter of many recent publications on singular boundary value problems, e.g., see [5, 14, 23, 27, 28, 33, 38]. Further, in the case of second-order boundary value problems, (1.1)–(1.4) occurs in applications involving nonlinear elliptic problems in annular regions, for this we refer to [8, 9, 22, 36]. Once again in all these applications, it is frequent that only solutions that are positive are useful.

Recently, several eigenvalue characterizations for particular cases of (1.1)–(1.4) have been carried out. To cite a few examples, Fink et al. [20] have dealt with the boundary value problem

$$\begin{aligned} &y'' + \lambda q(t) f(y) = 0, \quad t \in (0,1), \\ &y(0) = y(1) = 0. \end{aligned}$$

A more general problem, namely,

$$\begin{aligned} &y^{(n)} + \lambda q(t) f(y) = 0, \quad t \in (0,1), \\ &y^{(i)}(0) = y^{(n-2)}(1) = 0, \quad 0 \leqslant i \leqslant n-2 \end{aligned}$$

has been tackled in [11, 15]. Further, in [19] a different boundary value problem has been discussed

$$\begin{aligned} &y'' + \frac{N-1}{t} y' + \lambda q(t) f(y) = 0, \quad t \in (0,1), \\ &y'(0) = y(1) = 0. \end{aligned}$$

As for twin eigenfunctions, several studies on boundary value problems different from (1.1) (1.4) can be found in [7, 17, 30–32]. Our results not only generalize and extend the known theorems for all the above eigenvalue problems, but also complement the work of many authors [2–4, 16, 18, 26, 35, 37, 39–45], as well as include several other known criteria offered in [1].

The plan of the paper is as follows: In Section 2 we shall state a fixed point theorem given in [25, 29], and present some properties of certain Green's function which are needed later. By defining an appropriate Banach space and cone, in Section 3 we shall establish intervals of eigenvalues of (1.1)–(1.4). The investigation of twin eigenfunctions is carried out in Section 4. The boundary value problems (1.7), (1.3), (1.4) and (1.8), (1.3), (1.4) are treated, respectively, in Sections 5 and 6.

2. Preliminaries

Theorem 2.1 (Guo and Lakshmikantham [25], and Krasnosel'skii [29]). *Let B be a Banach space, and let $C \subset B$ be a cone in B. Assume Ω_1, Ω_2 are open subsets of B with $0 \in \Omega_1$, $\bar{\Omega}_1 \subset \Omega_2$, and let*

$$S : C \cap (\bar{\Omega}_2 \backslash \Omega_1) \to C$$

be a completely continuous operator such that, either
(a) $\|Sy\| \leqslant \|y\|$, $y \in C \cap \partial\Omega_1$, *and* $\|Sy\| \geqslant \|y\|$, $y \in C \cap \partial\Omega_2$, *or*
(b) $\|Sy\| \geqslant \|y\|$, $y \in C \cap \partial\Omega_1$, *and* $\|Sy\| \leqslant \|y\|$, $y \in C \cap \partial\Omega_2$.
Then, S has a fixed point in $C \cap (\bar{\Omega}_2 \backslash \Omega_1)$.

To obtain a solution for (1.1)–(1.4), we need a mapping whose kernel $g(t,s)$ is the Green's function of the boundary value problem

$$\begin{aligned} &-y^{(n)} = 0, \\ &y^{(i)}(0) = 0, \quad 0 \leqslant i \leqslant n-3, \\ &\alpha y^{(n-2)}(0) - \beta y^{(n-1)}(0) = 0, \\ &\gamma y^{(n-2)}(1) + \delta y^{(n-1)}(1) = 0. \end{aligned}$$

It can be verified that

$$G(t,s) = \frac{\partial^{n-2}}{\partial t^{n-2}} g(t,s)$$

is the Green's function of the boundary value problem

$$\begin{aligned} &-w'' = 0, \\ &\alpha w(0) - \beta w'(0) = 0, \\ &\gamma w(1) + \delta w'(1) = 0. \end{aligned}$$

Further, from [5] we have

$$G(t,s) = \frac{1}{\rho} \begin{cases} (\beta + \alpha s)[\delta + \gamma(1-t)], & 0 \leqslant s \leqslant t \\ (\beta + \alpha t)[\delta + \gamma(1-s)], & t \leqslant s \leqslant 1. \end{cases} \tag{2.1}$$

In view of conditions (1.5) and (1.6), it is clear that $G(t,s)$ is nonnegative on $[0,1] \times [0,1]$, and is positive on $(0,1) \times (0,1)$.

Lemma 2.2. *Let $m \in (0, \frac{1}{2})$. For $(t,s) \in [m, 1-m] \times [0,1]$, we have*

$$G(t,s) \geqslant K_m G(s,s), \tag{2.2}$$

where $0<K_m<1$ *is given by*

$$K_m = \min\left\{\frac{\delta+\gamma m}{\delta+\gamma}, \frac{\delta+\gamma(1-m)}{\delta+\gamma m}, \frac{\beta+\alpha m}{\beta+\alpha}, \frac{\beta+\alpha(1-m)}{\beta+\alpha m}\right\}. \tag{2.3}$$

Proof. For $0\leqslant s\leqslant t$, using (2.1), inequality (2.2) reduces to

$$\delta+\gamma(1-t)\geqslant K_m[\delta+\gamma(1-s)]. \tag{2.4}$$

In order that (2.4) holds, it is sufficient that K_m satisfies

$$\min_{t\in[m,1-m]}[\delta+\gamma(1-t)]\geqslant K_m \max_{s\in[0,1-m]}[\delta+\gamma(1-s)]. \tag{2.5}$$

If $\gamma\geqslant 0$, then (2.5) gives

$$\delta+\gamma[1-(1-m)]\geqslant K_m(\delta+\gamma) \quad \text{or} \quad K_m\leqslant\frac{\delta+\gamma m}{\delta+\gamma}. \tag{2.6}$$

If $\gamma<0$, then it follows from (2.5) that

$$\delta+\gamma(1-m)\geqslant K_m\{\delta+\gamma[1-(1-m)]\}, \quad \text{or} \quad K_m\leqslant\frac{\delta+\gamma(1-m)}{\delta+\gamma m}. \tag{2.7}$$

Next, for $t\leqslant s\leqslant 1$ the inequality (2.2) is the same as

$$\beta+\alpha t\geqslant K_m(\beta+\alpha s).$$

Again, it suffices to find K_m such that

$$\min_{t\in[m,1-m]}(\beta+\alpha t)\geqslant K_m \max_{s\in[m,1]}(\beta+\alpha s). \tag{2.8}$$

If $\alpha\geqslant 0$, then from (2.8) we obtain

$$\beta+\alpha m\geqslant K_m(\beta+\alpha), \quad \text{or} \quad K_m\leqslant\frac{\beta+\alpha m}{\beta+\alpha}. \tag{2.9}$$

If $\alpha<0$, then (2.8) yields

$$\beta+\alpha(1-m)\geqslant K_m(\beta+\alpha m), \quad \text{or} \quad K_m\leqslant\frac{\beta+\alpha(1-m)}{\beta+\alpha m}. \tag{2.10}$$

Combining (2.6), (2.7), (2.9) and (2.10), we immediately get

$$K_m\leqslant\min\left\{\frac{\delta+\gamma m}{\delta+\gamma}, \frac{\delta+\gamma(1-m)}{\delta+\gamma m}, \frac{\beta+\alpha m}{\beta+\alpha}, \frac{\beta+\alpha(1-m)}{\beta+\alpha m}\right\}.$$

The choice of K_m in (2.3) is now clear. □

Lemma 2.3 (Wong and Agarwal [37]). *For* $(t,s)\in[0,1]\times[0,1]$, *we have*

$$G(t,s)\leqslant LG(s,s) \tag{2.11}$$

where $L\geqslant 1$ *is given by*

$$L=\max\left\{1, \frac{\beta}{\beta+\alpha}, \frac{\delta}{\delta+\gamma}\right\}. \tag{2.12}$$

We introduce the following notations which are needed later: For a nonnegative y on $[0,1]$, we denote

$$\theta=\int_0^1 G(s,s)[q_1(s)-p(s)]f(y(s))\,\mathrm{d}s$$

and

$$\Gamma=\int_0^1 G(s,s)[q(s)-p_1(s)]f(y(s))\,\mathrm{d}s.$$

In view of (A1) and (A2), it is clear that $\theta\geqslant\Gamma>0$. Further, we define the constant

$$\xi=\frac{\Gamma}{L\theta}K_{1/4}.$$

It is noted that $0<\xi<1$.

3. Eigenvalue intervals of (1.1)–(1.4)

Let the Banach space

$$B=\{y\in C^{(n-2)}[0,1]\mid y^{(i)}(0)=0,\ 0\leqslant i\leqslant n-3\}$$

with norm $\|y\|=\sup_{t\in[0,1]}|y^{(n-2)}(t)|$, and let

$$C=\left\{y\in B\,\middle|\, y^{(n-2)}(t) \text{ is nonnegative on } [0,1];\ \min_{t\in[1/4,3/4]} y^{(n-2)}(t)\geqslant\xi\|y\|\right\}.$$

It is noted that C is a cone in B.

Lemma 3.1 (Wong and Agarwal [37]). *Let* $y\in B$. *For* $0\leqslant i\leqslant n-2$, *we have*

$$|y^{(i)}(t)|\leqslant\frac{t^{n-2-i}}{(n-2-i)!}\|y\|,\quad t\in[0,1]. \tag{3.1}$$

In particular,

$$|y(t)|\leqslant\frac{1}{(n-2)!}\|y\|,\quad t\in[0,1]. \tag{3.2}$$

Lemma 3.2 (Wong and Agarwal [37]). *Let* $y\in C$. *For* $0\leqslant i\leqslant n-2$, *we have*

$$y^{(i)}(t)\geqslant 0,\quad t\in[0,1], \tag{3.3}$$

and

$$y^{(i)}(t) \geqslant \left(t-\frac{1}{4}\right)^{n-2-i} \frac{\xi}{(n-2-i)!}\|y\|, \quad t \in \left[\frac{1}{4}, \frac{3}{4}\right]. \tag{3.4}$$

In particular,

$$y(t) \geqslant \frac{\xi}{4^{n-2}(n-2)!}\|y\|, \quad t \in \left[\frac{1}{2}, \frac{3}{4}\right]. \tag{3.5}$$

Remark 3.3. If $y \in C$ is a nontrivial solution of (1.1)–(1.4), then (3.3) and (3.5) imply that y is a positive solution of (1.1)–(1.4).

To obtain a positive solution of (1.1)–(1.4), we shall seek a fixed point of the operator λS in the cone C, where $S: C \to B$ is defined by

$$Sy(t) = \int_0^1 g(t,s)[Q(s,y,y',\ldots,y^{(n-2)}) - P(s,y,y',\ldots,y^{(n-2)})]\,\mathrm{d}s, \quad t \in [0,1]. \tag{3.6}$$

It follows that

$$(Sy)^{(n-2)}(t) = \int_0^1 G(t,s)[Q(s,y,y',\ldots,y^{(n-2)}) - P(s,y,y',\ldots,y^{(n-2)})]\,\mathrm{d}s, \quad t \in [0,1].$$

In view of condition (A1), we get for $t \in [0,1]$,

$$\begin{aligned} \int_0^1 G(t,s)[q(s) - p_1(s)]f(y(s))\,\mathrm{d}s &\leqslant (Sy)^{(n-2)}(t) \\ &\leqslant \int_0^1 G(t,s)[q_1(s) - p(s)]f(y(s))\,\mathrm{d}s. \end{aligned} \tag{3.7}$$

We shall now show that the operator λS maps C into itself. For this, let $y \in C$. From (3.7) and (A2), we find

$$(\lambda S y)^{(n-2)}(t) \geqslant \lambda \int_0^1 G(t,s)[q(s) - p_1(s)]f(y(s))\,\mathrm{d}s \geqslant 0, \quad t \in [0,1]. \tag{3.8}$$

Further, it follows from (3.7) and Lemma 2.3 that

$$\begin{aligned} (Sy)^{(n-2)}(t) &\leqslant \int_0^1 G(t,s)[q_1(s) - p(s)]f(y(s))\,\mathrm{d}s \\ &\leqslant L \int_0^1 G(s,s)[q_1(s) - p(s)]f(y(s))\,\mathrm{d}s = L\theta, \quad t \in [0,1]. \end{aligned}$$

Therefore,

$$\|Sy\| \leqslant L\theta \quad \text{or} \quad 1 \geqslant \frac{\|Sy\|}{L\theta}. \tag{3.9}$$

Now, on using (3.7), Lemma 2.2 and (3.9), we find for $t\in[\frac{1}{4},\frac{3}{4}]$,

$$\begin{aligned}(\lambda Sy)^{(n-2)}(t) &\geqslant \lambda\int_0^1 G(t,s)[q(s)-p_1(s)]f(y(s))\,ds\\ &\geqslant \lambda K_{1/4}\int_0^1 G(s,s)[q(s)-p_1(s)]f(y(s))\,ds\\ &= \lambda\Gamma K_{1/4}\geqslant \lambda\Gamma K_{1/4}\frac{\|Sy\|}{L\theta}=\lambda\xi\|Sy\|=\xi\|\lambda Sy\|.\end{aligned}$$

Subsequently,

$$\min_{t\in[1/4,3/4]}(\lambda Sy)^{(n-2)}(t)\geqslant \xi\|\lambda Sy\|. \tag{3.10}$$

It follows from (3.8) and (3.10) that $\lambda Sy\in C$. Hence, $(\lambda S)(C)\subseteq C$. Also, by standard arguments λS is completely continuous.

In the following results, we shall use the number $t^*\in[0,1]$ which is defined by

$$\int_{1/2}^{3/4} G(t^*,s)[q(s)-p_1(s)]\,ds=\sup_{t\in[0,1]}\int_{1/2}^{3/4} G(t,s)[q(s)-p_1(s)]\,ds. \tag{3.11}$$

Theorem 3.4. *Suppose that $f_0\in[0,\infty)$ and $f_\infty\in(0,\infty)$. Then, for each λ satisfying*

$$\frac{1}{\xi\mu f_\infty}<\lambda<\frac{1}{L\nu f_0}, \tag{3.12}$$

where

$$\mu=\frac{1}{4^{n-2}(n-2)!}\int_{1/2}^{3/4} G(t^*,s)[q(s)-p_1(s)]\,ds \tag{3.13}$$

and

$$\nu=\frac{1}{(n-2)!}\int_{1/2}^{3/4} s^{n-2}G(s,s)[q_1(s)-p(s)]\,ds, \tag{3.14}$$

the boundary value problem (1.1)–(1.4) *has a positive solution.*

Proof. Let λ satisfy (3.12). Then, we may choose $\varepsilon>0$ such that

$$\frac{1}{\xi\mu(f_\infty-\varepsilon)}\leqslant\lambda\leqslant\frac{1}{L\nu_1(f_0+\varepsilon)} \tag{3.15}$$

where

$$\nu_1=\frac{1}{(n-2)!}\int_0^1 s^{n-2}G(s,s)[q_1(s)-p(s)]\,ds.$$

Since $f_0\in[0,\infty)$, we let $c>0$ be such that

$$f(u)\leqslant(f_0+\varepsilon)u,\quad 0<u\leqslant c. \tag{3.16}$$

Let $y \in C$ be such that $\|y\| = (n-2)!c$. Then, by (3.2) we have $y(t) \leqslant c$, $t \in [0,1]$, Applying (3.7), Lemma 2.3, (3.16), (3.1) and (3.15) successively, we find for $t \in [0,1]$,

$$(\lambda S y)^{(n-2)}(t) \leqslant \lambda L \int_0^1 G(s,s)[q_1(s) - p(s)] f(y(s))\,\mathrm{d}s$$

$$\leqslant \lambda L \int_0^1 G(s,s)[q_1(s) - p(s)](f_0 + \varepsilon) y(s)\,\mathrm{d}s$$

$$\leqslant \lambda L \int_0^1 G(s,s)[q_1(s) - p(s)](f_0 + \varepsilon) \frac{s^{n-2}}{(n-2)!} \|y\|\,\mathrm{d}s \leqslant \|y\|.$$

Hence,

$$\|\lambda S y\| \leqslant \|y\|. \tag{3.17}$$

If we set $\Omega_1 = \{y \in B \mid \|y\| < (n-2)!c\}$, then (3.17) holds for $y \in C \cap \partial\Omega_1$.

Next, since $f_\infty \in (0,\infty)$, we may choose $d > 0$ such that

$$f(u) \geqslant (f_\infty - \varepsilon)u, \quad u \geqslant d. \tag{3.18}$$

Let $y \in C$ be such that $\|y\| = d' \equiv \max\{2(n-2)!c, 4^{n-2}(n-2)!d/\xi\}$. Then, from (3.5) we have

$$y(t) \geqslant \frac{\xi}{4^{n-2}(n-2)!}\|y\| \geqslant \frac{\xi}{4^{n-2}(n-2)!} \cdot 4^{n-2}(n-2)!\frac{d}{\xi} = d, \quad t \in \left[\frac{1}{2}, \frac{3}{4}\right].$$

In view of (3.18), this leads to

$$f(y(t)) \geqslant (f_\infty - \varepsilon) y(t), \quad t \in [\tfrac{1}{2}, \tfrac{3}{4}]. \tag{3.19}$$

Using (3.7), (3.19), (3.5) and (3.15), we find

$$(\lambda S y)^{(n-2)}(t^*) \geqslant \lambda \int_0^1 G(t^*,s)\,[q(s) - p_1(s)]\, f(y(s))\,\mathrm{d}s$$

$$\geqslant \lambda \int_{1/2}^{3/4} G(t^*,s)\,[q(s) - p_1(s)]\, f(y(s))\,\mathrm{d}s$$

$$\geqslant \lambda \int_{1/2}^{3/4} G(t^*,s)\,[q(s) - p_1(s)]\,(f_\infty - \varepsilon) y(s)\,\mathrm{d}s$$

$$\geqslant \lambda \int_{1/2}^{3/4} G(t^*,s)[q(s) - p_1(s)](f_\infty - \varepsilon)\frac{\xi}{4^{n-2}(n-2)!}\|y\|\,\mathrm{d}s$$

$$\geqslant \|y\|.$$

Therefore,

$$\|\lambda S y\| \geqslant \|y\|. \tag{3.20}$$

By setting $\Omega_2 = \{y \in B \mid \|y\| < d'\}$, we see that (3.20) holds for $y \in C \cap \partial\Omega_2$.

Now that we have obtained (3.17) and (3.20), it follows from Theorem 2.1 that λS has a fixed point $y \in C \cap (\bar{\Omega}_2 \backslash \Omega_1)$ such that

$$(n-2)!c \leqslant \|y\| \leqslant d'.$$

It is clear that this y is a positive solution of (1.1)–(1.4). □

The following corollary is immediate from Theorem 3.4.

Corollary 3.5. *Suppose that* $f_0 \in [0,\infty)$ *and* $f_\infty \in (0,\infty)$. *Then,*

$$\left(\frac{1}{\xi\mu f_\infty}, \frac{1}{Lv f_0}\right) \subseteq E,$$

where μ *and* v *are defined in Theorem* 3.4.

Theorem 3.6. *Suppose that* $f_0 \in (0,\infty)$ *and* $f_\infty \in [0,\infty)$. *Then, for each* λ *satisfying*

$$\frac{1}{\xi\mu f_0} < \lambda < \frac{1}{Lv f_\infty}, \tag{3.21}$$

where μ *and* v *are defined in Theorem* 3.4, *the boundary value problem* (1.1)–(1.4) *has a positive solution.*

Proof. Let λ fulfill (3.21). Then, we may choose $\varepsilon > 0$ so that

$$\frac{1}{\xi\mu(f_0-\varepsilon)} \leqslant \lambda \leqslant \frac{1}{Lv_2(f_\infty+\varepsilon)}, \tag{3.22}$$

where

$$v_2 = \frac{1}{(n-2)!}\int_0^1 G(s,s)[q_1(s)-p(s)]\,\mathrm{d}s.$$

Since $f_0 \in (0,\infty)$, there exists $\bar{c} > 0$ such that

$$f(u) \geqslant (f_0-\varepsilon)u, \quad 0 < u \leqslant \bar{c}. \tag{3.23}$$

Let $y \in C$ be such that $\|y\| = (n-2)!\bar{c}$. Then, it follows from (3.2) that $y(t) \leqslant \bar{c}, t \in [0,1]$. On using (3.7), (3.23), (3.5) and (3.22) successively, we get

$$\begin{aligned}
(\lambda S y)^{(n-2)}(t^*) &\geqslant \lambda \int_{1/2}^{3/4} G(t^*,s)[q(s)-p_1(s)]f(y(s))\,\mathrm{d}s \\
&\geqslant \lambda \int_{1/2}^{3/4} G(t^*,s)[q(s)-p_1(s)](f_0-\varepsilon)y(s)\,\mathrm{d}s \\
&\geqslant \lambda \int_{1/2}^{3/4} G(t^*,s)[q(s)-p_1(s)](f_0-\varepsilon)\frac{\xi}{4^{n-2}(n-2)!}\|y\|\,\mathrm{d}s \\
&\geqslant \|y\|
\end{aligned}$$

from which inequality (3.20) is immediate. By setting $\Omega_1=\{y\in B \mid \|y\|<(n-2)!\bar{c}\}$, we see that (3.20) holds for $y\in C\cap\partial\Omega_1$.

Next, noting that $f_\infty\in[0,\infty)$, we may choose $\bar{d}>0$ such that

$$f(u)\leqslant(f_\infty+\varepsilon)u, \quad u\geqslant\bar{d}. \tag{3.24}$$

There are two cases to consider, namely, f is bounded and f is unbounded.

Case 1: Suppose that f is bounded, i.e., there exists some $R>0$ such that

$$f(u)\leqslant R, \quad u\in(0,\infty). \tag{3.25}$$

Define

$$d_1=\max\left\{2(n-2)!\bar{c}, \frac{\lambda LR}{(n-2)!}\int_0^1 G(s,s)[q_1(s)-p(s)]\,\mathrm{d}s\right\}.$$

Let $y\in C$ be such that $\|y\|=(n-2)!d_1$. From (3.7), Lemma 2.3 and (3.25), we find for $t\in[0,1]$,

$$\begin{aligned}(\lambda Sy)^{(n-2)}(t) &\leqslant \lambda L\int_0^1 G(s,s)[q_1(s)-p(s)]f(y(s))\,\mathrm{d}s\\ &\leqslant \lambda L\int_0^1 G(s,s)[q_1(s)-p(s)]R\,\mathrm{d}s\\ &\leqslant (n-2)!d_1=\|y\|.\end{aligned}$$

Hence, (3.17) holds.

Case 2: Suppose that f is unbounded. Then, there exists $d_1>\max\left\{2(n-2)!\bar{c},\ \bar{d}\right\}$ such that

$$f(u)\leqslant f(d_1), \quad 0<u\leqslant d_1. \tag{3.26}$$

Let $y\in C$ be such that $\|y\|=(n-2)!d_1$. Then, by (3.2) we have $y(t)\leqslant d_1, t\in[0,1]$. Applying (3.7), Lemma 2.3, (3.26), (3.24) and (3.22) successively, we get

$$\begin{aligned}(\lambda Sy)^{(n-2)}(t) &\leqslant \lambda L\int_0^1 G(s,s)[q_1(s)-p(s)]f(y(s))\,\mathrm{d}s\\ &\leqslant \lambda L\int_0^1 G(s,s)[q_1(s)-p(s)]f(d_1)\,\mathrm{d}s\\ &\leqslant \lambda L\int_0^1 G(s,s)[q_1(s)-p(s)](f_\infty+\varepsilon)d_1\,\mathrm{d}s\\ &= \lambda L\int_0^1 G(s,s)[q_1(s)-p(s)](f_\infty+\varepsilon)\frac{\|y\|}{(n-2)!}\,\mathrm{d}s\\ &\leqslant \|y\|, \quad t\in[0,1]\end{aligned}$$

from which (3.17) follows immediately.

In both Cases 1 and 2, if we set $\Omega_2=\{y\in B \mid \|y\|<(n-2)!d_1\}$, then (3.17) holds for $y\in C\cap\partial\Omega_2$.

Having obtained both (3.20) and (3.17), it follows from Theorem 2.1 that λS has a fixed point $y \in C \cap \left(\bar{\Omega}_2 \backslash \Omega_1\right)$ such that

$$(n-2)!\bar{c} \leqslant \|y\| \leqslant (n-2)!d_1.$$

Clearly, this y is a positive solution of (1.1)–(1.4). □

Theorem 3.6 leads to the following corollary.

Corollary 3.7. *Suppose that* $f_0 \in (0,\infty)$ *and* $f_\infty \in [0,\infty)$. *Then,*

$$\left(\frac{1}{\xi\mu f_0}, \frac{1}{Lv f_\infty}\right) \subseteq E$$

where μ *and* v *are defined in Theorem* 3.4.

Example 3.8. Consider the boundary value problem

$$y'' + \frac{\lambda}{|\sin 6(t-t^2+2)|}|\sin 6y| = 0, \quad t \in (0,1),$$

$$y(0) - 2y'(0) = 0, \quad y(1) + 2y'(1) = 0.$$

Taking $f(y) = |\sin 6y|$, we find

$$\frac{Q(t,y)}{f(y)} = \frac{1}{|\sin 6(t-t^2+2)|} \quad \text{and} \quad \frac{P(t,y)}{f(y)} = 0.$$

Hence, we may take

$$q(t) = q_1(t) = \frac{1}{|\sin 6(t-t^2+2)|} \quad \text{and} \quad p(t) = p_1(t) = 0.$$

All the hypotheses (A1) and (A2) are satisfied.

Clearly, $f_0 = 6$ and $f_\infty = 0$. Using (2.1), by direct computation we have

$$\mu = \int_{1/2}^{3/4} \frac{G(t^*,s)}{|\sin 6(s-s^2+2)|}\,\mathrm{d}s = \int_{1/2}^{3/4} \frac{G(0.631,s)}{|\sin 6(s-s^2+2)|}\,\mathrm{d}s = 0.428.$$

Hence, it follows from Corollary 3.7 that

$$\left(\frac{1}{\xi\mu f_0}, \frac{1}{Lv f_\infty}\right) = (0.519, \infty) \subseteq E.$$

In fact, when $\lambda = 2$, the eigenfunction is given by $y(t) = t(1-t) + 2$.

Example 3.9. Consider the boundary value problem

$$y^{(3)} + \frac{\lambda t(0.1y + 1 - \mathrm{e}^{-20y})}{0.1(7t-t^2-t^4) + 1 - \exp(-20(7t-t^2-t^4))} = 0, \quad t \in (0,1),$$

$$y(0) = 0, \quad -2y'(0) - 7y''(0) = 0, \quad 14y'(1) + y''(1) = 0.$$

Choosing $f(y)=0.1y+1-e^{-20y}$, we have $f_0=20.1$ and $f_\infty=0.1$. If we take

$$q(t)=q_1(t)=\frac{t}{0.1(7t-t^2-t^4)+1-\exp(-20(7t-t^2-t^4))}$$

and $p(t)=p_1(t)=0$, then it can be computed that $\mu=0.0179(t^*=0)$ and $\nu=0.0364$. Thus, we conclude from Corollary 3.7 that $(13.0,\ 196)\subseteq E$. Indeed, when $\lambda=24$, the eigenfunction is given by $y(t)=t(7-t-t^3)$.

Example 3.10. Consider the boundary value problem

$$y^{(4)}+\frac{\lambda(0.1y+\tanh 180y)}{0.1(9t^2-t^3-t^4)+\tanh 180(9t^2-t^3-t^4)}=0,\quad t\in(0,1),$$

$$y(0)=y'(0)=y''(1)=0,\quad -y''(0)-3y^{(3)}(0)=0.$$

With $f(y)=0.1y+\tanh 180y$, we have $f_0=180.1$ and $f_\infty=0.1$. Further, by taking

$$q(t)=q_1(t)=[0.1(9t^2-t^3-t^4)+\tanh 180(9t^2-t^3-t^4)]^{-1}$$

and $p(t)=p_1(t)=0$, we compute that $\mu=7.09\cdot 10^{-3}(t^*=0)$ and $\nu=0.0160$. Therefore, it follows from Corollary 3.7 that $(4.70,\ 417)\subseteq E$. As an example, when $\lambda=24$, the eigenfunction is given by $y(t)=t^2(9-t-t^2)$.

4. Existence of twin eigenfunctions

Throughout this section, we let $\lambda=1$ in (1.1).

Theorem 4.1. *Let $r>0$ be given. Suppose that f satisfies*

$$0<f(u)\leqslant(n-2)!r\left\{L\int_0^1 G(s,s)[q_1(s)-p(s)]\,ds\right\}^{-1},\quad 0<u\leqslant r \tag{4.1}$$

and

$$f_0=\infty. \tag{4.2}$$

Then, the boundary value problem (1.1)–(1.4) *has an eigenfunction y such that*

$$0<\|y\|\leqslant(n-2)!r. \tag{4.3}$$

Proof. Since $f_0=\infty$, there exist $J>0$ and $0<c<r$ such that

$$f(u)\geqslant Ju,\quad 0<u\leqslant c \tag{4.4}$$

and

$$\frac{J\xi}{4^{n-2}(n-2)!}\int_{1/2}^{3/4} G\left(\frac{1}{2},s\right)[q(s)-p_1(s)]\,ds\geqslant 1. \tag{4.5}$$

Let $y \in C$ be such that $\|y\| = (n-2)!c$. By (3.2), we have $y(t) \leqslant c$, $t \in [0,1]$. On using (3.7), (4.4), (3.5) and (4.5) successively, we get

$$\begin{aligned}
(Sy)^{(n-2)}\left(\frac{1}{2}\right) &\geqslant \int_0^1 G\left(\frac{1}{2}, s\right)[q(s) - p_1(s)]f(y(s))\,\mathrm{d}s \\
&\geqslant \int_0^1 G\left(\frac{1}{2}, s\right)[q(s) - p_1(s)]Jy(s)\,\mathrm{d}s \\
&\geqslant \int_{1/2}^{3/4} G\left(\frac{1}{2}, s\right)[q(s) - p_1(s)]Jy(s)\,\mathrm{d}s \\
&\geqslant \int_{1/2}^{3/4} G\left(\frac{1}{2}, s\right)[q(s) - p_1(s)]J\frac{\xi}{4^{n-2}(n-2)!}\|y\|\,\mathrm{d}s \geqslant \|y\|.
\end{aligned}$$

This immediately implies that

$$\|Sy\| \geqslant \|y\|. \tag{4.6}$$

If we set $\Omega_1 = \{y \in B \mid \|y\| < (n-2)!c\}$, then (4.6) holds for $y \in C \cap \partial\Omega_1$.

Next, let $y \in C$ be such that $\|y\| = (n-2)!r$. Once again, by (3.2) this leads to $y(t) \leqslant r$, $t \in [0,1]$. Then, in view of (3.7), Lemma 2.3 and (4.1), we find

$$(Sy)^{(n-2)}(t) \leqslant L \int_0^1 G(s,s)[q_1(s) - p(s)]f(y(s))\,\mathrm{d}s \leqslant (n-2)!r = \|y\|, \quad t \in [0,1].$$

Hence,

$$\|Sy\| \leqslant \|y\|. \tag{4.7}$$

By setting $\Omega_2 = \{y \in B \mid \|y\| < (n-2)!r\}$, we see that (4.7) holds for $y \in C \cap \partial\Omega_2$.

Having obtained (4.6) and (4.7), it follows from Theorem 2.1 that S has a fixed point $y \in C \cap (\bar{\Omega}_2 \backslash \Omega_1)$ such that

$$(n-2)!c \leqslant \|y\| \leqslant (n-2)!r.$$

Clearly, this y is a positive solution of (1.1)–(1.4) that fulfills (4.3). $\square$

Theorem 4.2. *Let $r > 0$ be given. Suppose that f satisfies condition* (4.1) *and*

$$f_\infty = \infty. \tag{4.8}$$

Then, the boundary value problem (1.1)–(1.4) *has an eigenfunction y such that*

$$\|y\| \geqslant (n-2)!r. \tag{4.9}$$

Proof. As in the proof of Theorem 4.1, condition (4.1) gives rise to (4.7). If we set $\Omega_1 = \{y \in B \mid \|y\| < (n-2)!r\}$, then (4.7) holds for $y \in C \cap \partial\Omega_1$.

Next, since $f_\infty = \infty$, we may choose $N > 0$ and $d > r$ such that

$$f(u) \geqslant Nu, \quad u \geqslant d \tag{4.10}$$

and

$$\frac{N\xi}{4^{n-2}(n-2)!} \int_{1/2}^{3/4} G\left(\frac{1}{2}, s\right) [q(s) - p_1(s)]\, ds \geqslant 1. \tag{4.11}$$

Let $y \in C$ be such that $\|y\| = 4^{n-2}(n-2)!d/\xi$. Then, from (3.5) we have for $t \in [\frac{1}{2}, \frac{3}{4}]$,

$$y(t) \geqslant \frac{\xi}{4^{n-2}(n-2)!}\|y\| = \frac{\xi}{4^{n-2}(n-2)!} \cdot 4^{n-2}(n-2)!\frac{d}{\xi} = d,$$

which in view of (4.10) leads to

$$f(y(t)) \geqslant Ny(t), \quad t \in [\tfrac{1}{2}, \tfrac{3}{4}]. \tag{4.12}$$

Using (3.7), (4.12), (3.5) and (4.11), we find

$$\begin{aligned}(Sy)^{(n-2)}\left(\frac{1}{2}\right) &\geqslant \int_{1/2}^{3/4} G\left(\frac{1}{2}, s\right) [q(s) - p_1(s)] f(y(s))\, ds \\ &\geqslant \int_{1/2}^{3/4} G\left(\frac{1}{2}, s\right) [q(s) - p_1(s)] N y(s)\, ds \\ &\geqslant \int_{1/2}^{3/4} G\left(\frac{1}{2}, s\right) [q(s) - p_1(s)] N \frac{\xi}{4^{n-2}(n-2)!}\|y\|\, ds \geqslant \|y\|.\end{aligned}$$

Therefore, (4.6) holds. If we set $\Omega_2 = \{y \in B \mid \|y\| < 4^{n-2}(n-2)!d/\xi\}$, then (4.6) holds for $y \in C \cap \partial\Omega_2$.

Now that we have obtained (4.7) and (4.6), it follows from Theorem 2.1 that S has a fixed point $y \in C \cap (\bar{\Omega}_2 \backslash \Omega_1)$ such that

$$(n-2)!r \leqslant \|y\| \leqslant 4^{n-2}(n-2)!\frac{d}{\xi}.$$

It is clear that this y is a positive solution of (1.1)–(1.4) satisfying (4.9). □

Theorem 4.3. *Let $r > 0$ be given. Suppose that f satisfies conditions* (4.1), (4.2) *and* (4.8). *Then, the boundary value problem* (1.1)–(1.4) *has twin eigenfunctions y_1 and y_2 such that*

$$0 < \|y_1\| \leqslant (n-2)!r \leqslant \|y_2\|. \tag{4.13}$$

Proof. This is a direct consequence of Theorems 4.1 and 4.2. □

The following example illustrates Theorem 4.3.

Example 4.4. Consider the boundary value problem

$$y^{(3)} + \frac{576t}{(7t-t^2-t^4)^2+576M}(y^2+M)=0, \quad t\in(0,1),$$

$$y(0)=0, \quad -2y'(0)-7y''(0)=0, \quad 14y'(1)+y''(1)=0$$

where $M>0$.

Taking $f(y)=y^2+M$, we may choose

$$q(t)=q_1(t)=\frac{576t}{(7t-t^2-t^4)^2+576M} \quad \text{and} \quad p(t)=p_1(t)=0.$$

It is obvious that f satisfies (4.2) and (4.8). Since

$$f(u)\leqslant r^2+M, \quad 0<u\leqslant r,$$

in order that the condition (4.1) is fulfilled, we set

$$r^2+M\leqslant(n-2)!r\left\{L\int_0^1 G(s,s)[q_1(s)-p(s)]\mathrm{d}s\right\}^{-1}=r\left[\frac{7}{5}\int_0^1 G(s,s)q_1(s)\,\mathrm{d}s\right]^{-1}. \tag{4.14}$$

Case 1: $M=0.45$. Solving the quadratic inequality (4.14), we get

$$0.589\leqslant r\leqslant 0.764. \tag{4.15}$$

Hence, (4.1) holds for any $r\in[0.589, 0.764]$. By Theorem 4.3, the boundary value problem has twin eigenfunctions y_1 and y_2 such that (4.13) holds. In view of (4.15), it is clear that

$$0<\|y_1\|\leqslant 0.589 \quad \text{and} \quad \|y_2\|\geqslant 0.764. \tag{4.16}$$

Case 2: $M=1$. Here, the quadratic inequality (4.14) provides

$$0.393\leqslant r\leqslant 2.54. \tag{4.17}$$

Once again, by Theorem 4.3 the boundary value problem has twin eigenfunctions y_1 and y_2 satisfying (4.13). It now follows from (4.17) that

$$0<\|y_1\|\leqslant 0.393 \quad \text{and} \quad \|y_2\|\geqslant 2.54. \tag{4.18}$$

Indeed, for both cases an eigenfunction is given by $y(t)=\frac{1}{24}t(7-t-t^3)$. We note that $\|y\|=\sup_{t\in[0,1]}|y'(t)|=y'(0)=0.292$ is within the ranges obtained in (4.16) and (4.18).

5. Twin eigenfunctions of (1.7), (1.3), (1.4)

In this section as well as in Section 6, we have $n=2$ and $\|y\|=\sup_{t\in[0,1]}|y(t)|$.

Theorem 5.1. *Let $r>0$ be given. Suppose that*

$$L\int_0^1 G(s,s)h(s)\,\mathrm{d}s \leqslant \frac{r}{r^a+r^b}. \tag{5.1}$$

Then, the boundary value problem (1.7), (1.3), (1.4) *has twin eigenfunctions y_1 and y_2 such that*

$$0<\|y_1\|\leqslant r\leqslant \|y_2\|. \tag{5.2}$$

Proof. Let $f(u)=u^a+u^b$. Then, f satisfies (4.2) and (4.8). Further, we may take

$$q(t)=q_1(t)=h(t) \quad \text{and} \quad p(t)=p_1(t)=0.$$

Clearly,

$$f(u)\leqslant r^a+r^b, \quad 0<u\leqslant r.$$

So, to ensure that (4.1) is satisfied, we impose

$$r^a+r^b\leqslant r\left\{L\int_0^1 G(s,s)[q_1(s)-p(s)]\,\mathrm{d}s\right\}^{-1}$$

which is exactly (5.1).

The conclusion now follows from Theorem 4.3. □

Remark 5.2. In [44] we have discussed two particular cases of (1.7), (1.3), (1.4).

Case 1: $\alpha=\gamma=1$, $\beta=\delta=0$. The condition corresponding to (5.1) is obtained [44] as

$$\int_0^1 (1-s)h(s)\,\mathrm{d}s\leqslant \frac{r}{r^a+r^b}. \tag{5.3}$$

It is noted that for this special case (5.1) reduces to

$$\int_0^1 s(1-s)h(s)\,\mathrm{d}s\leqslant \frac{r}{r^a+r^b} \tag{5.4}$$

which is clearly a weaker condition than (5.3), and hence is an improvement over (5.3).

Case 2: $\alpha=\delta=1$, $\beta=\gamma=0$. In [44], the condition corresponding to (5.1) is

$$\int_0^1 h(s)\,\mathrm{d}s\leqslant \frac{r}{r^a+r^b}. \tag{5.5}$$

We note that in this case (5.1) is the same as

$$\int_0^1 sh(s)\,\mathrm{d}s\leqslant \frac{r}{r^a+r^b} \tag{5.6}$$

which is less restrictive than (5.5), and therefore is an improvement over (5.5).

Example 5.3. Consider the boundary value problem

$$y'' + h(t)(y^a + y^b) = 0, \quad t \in (0,1),$$
$$y(0) - 2y'(0) = 0, \quad -y(1) + 7y'(1) = 0,$$

where $0 \leqslant a < 1 < b$.

Let $r = 1$ be given. Then, condition (5.1) reduces to

$$\int_0^1 (2+s)(6+s)h(s)\,\mathrm{d}s \leqslant \tfrac{12}{7}. \tag{5.7}$$

By Theorem 5.1, for any $h(t)$ that fulfills (5.7), the boundary value problem has twin eigenfunctions y_1 and y_2 such that

$$0 < \|y_1\| \leqslant 1 \leqslant \|y_2\|.$$

Some examples of such $h(t)$ are $h(t) = \frac{1}{8}t,\ \frac{1}{10}\sin \pi t,\ 1/7(t+1)$.

Now, we shall establish upper and lower bounds for the twin eigenfunctions.

Theorem 5.4. *We define*

$$\phi(u) = \frac{2\delta+\gamma}{2\rho} \sup_{m\in(0,1/2)} \left(\frac{K_m}{L}\right)^u (\beta + \alpha m)(1-2m)h^*(m)$$

where

$$h^*(m) = \min_{t\in[m,1-m]} h(t), \tag{5.8}$$
$$w = [\phi(a)]^{-1/(a-1)} \quad \textit{and} \quad v = [\phi(b)]^{-1/(b-1)}.$$

Let $r > 0$ be given. Suppose that (5.1) *holds. Then, the boundary value problem* (1.7), (1.3), (1.4) *has twin eigenfunctions y_1 and y_2 such that*

(a) *if* $r < \min\{w, v\}$, $0 < \|y_1\| \leqslant r \leqslant \|y_2\| \leqslant \min\{w, v\}$;

(b) *if* $\min\{w, v\} < r < \max\{w, v\}$, $\min\{w, v\} \leqslant \|y_1\| \leqslant r \leqslant \|y_2\| \leqslant \max\{w, v\}$;

(c) *if* $r > \max\{w, v\}$, $\max\{w, v\} \leqslant \|y_1\| \leqslant r \leqslant \|y_2\|$.

Proof. Since (5.1) is satisfied, it follows from Theorem 5.1 that (1.7), (1.3), (1.4) has twin eigenfunctions y_1 and y_2 such that (5.2) holds.

To establish upper and lower bounds for the two eigenfunctions, for an arbitrary $m \in (0, \frac{1}{2})$, we let C_m be a cone in B defined by

$$C_m = \left\{ y \in B \;\middle|\; y(t) \text{ is nonnegative on } [0,1];\ \min_{t\in[m,1-m]} y(t) \geqslant \frac{K_m}{L}\|y\| \right\}. \tag{5.9}$$

Define the operator $S : C_m \to B$ by

$$Sy(t) = \int_0^1 G(t,s)h(s)[y(s)^a + y(s)^b]\,\mathrm{d}s, \quad t \in [0,1].$$

To obtain an eigenfunction of (1.7), (1.3), (1.4), we shall seek a fixed point of S in the cone C_m.

We shall show that S maps C_m into itself. For this, let $y\in C_m$. It is clear that $Sy(t)$ is nonnegative on $[0,1]$. Further, by Lemma 2.3 we have

$$Sy(t)\leqslant L\int_0^1 G(s,s)h(s)[y(s)^a+y(s)^b]\,\mathrm{d}s,\quad t\in[0,1]$$

which implies

$$\|Sy\|\leqslant L\int_0^1 G(s,s)h(s)[y(s)^a+y(s)^b]\,\mathrm{d}s. \tag{5.10}$$

Next, for $t\in[m,1-m]$, it follows from Lemma 2.2 and (5.10) that

$$Sy(t)\geqslant K_m\int_0^1 G(s,s)h(s)[y(s)^a+y(s)^b]\,\mathrm{d}s\geqslant K_m\frac{\|Sy\|}{L}.$$

Thus,

$$\min_{t\in[m,1-m]} Sy(t)\geqslant \frac{K_m}{L}\|Sy\|$$

and so $Sy\in C_m$. Also, the standard arguments yield that S is completely continuous.

Let $y\in C_m$ be such that $\|y\|=r$. In view of Lemma 2.3 and (5.1), we find

$$\begin{aligned}Sy(t)&\leqslant L\int_0^1 G(s,s)h(s)[y(s)^a+y(s)^b]\,\mathrm{d}s\\&\leqslant L\int_0^1 G(s,s)h(s)(r^a+r^b)\,\mathrm{d}s\leqslant r=\|y\|,\quad t\in[0,1].\end{aligned}$$

Therefore,

$$\|Sy\|\leqslant\|y\|. \tag{5.11}$$

If we set $\Omega=\{y\in B\mid \|y\|<r\}$, then (5.11) holds for $y\in C_m\cap\partial\Omega$.

Now, let $y\in C_m$. It follows that

$$\begin{aligned}\|Sy\|&=\sup_{t\in[0,1]}\int_0^1 G(t,s)h(s)[y(s)^a+y(s)^b]\,\mathrm{d}s\\&\geqslant\int_0^1 G(m,s)h(s)[y(s)^a+y(s)^b]\,\mathrm{d}s\\&\geqslant\int_m^{1-m} G(m,s)h^*(m)[y(s)^a+y(s)^b]\,\mathrm{d}s\\&\geqslant\int_m^{1-m} G(m,s)h^*(m)\left[\left(\frac{K_m}{L}\right)^a\|y\|^a+\left(\frac{K_m}{L}\right)^b\|y\|^b\right]\mathrm{d}s.\end{aligned}$$

On substituting

$$G(m,s)=\frac{1}{\rho}(\beta+\alpha m)[\delta+\gamma(1-s)],\quad s\in[m,1] \tag{5.12}$$

into the above inequality, we simplify and then take supremum over m to obtain

$$\|Sy\| \geqslant \phi(a)\|y\|^a + \phi(b)\|y\|^b. \tag{5.13}$$

Let $y \in C_m$ be such that $\|y\| = w$. Then, (5.13) provides

$$\|Sy\| \geqslant \phi(a)\|y\|^a = \phi(a)\|y\|^{a-1}\|y\| = \|y\|. \tag{5.14}$$

If we set $\Omega_1 = \{y \in B \mid \|y\| < w\}$, then (5.14) holds for $y \in C_m \cap \partial\Omega_1$. Now that we have obtained (5.11) and (5.14), it follows from Theorem 2.1 that S has a fixed point y_1 such that

$$\min\{w, r\} \leqslant \|y_1\| \leqslant \max\{w, r\}. \tag{5.15}$$

Likewise, if we let $y \in C_m$ be such that $\|y\| = v$, then from (5.13) we get

$$\|Sy\| \geqslant \phi(b)\|y\|^b = \phi(b)\|y\|^{b-1}\|y\| = \|y\|. \tag{5.16}$$

By setting $\Omega_2 = \{y \in B \mid \|y\| < v\}$, we see that (5.16) holds for $y \in C_m \cap \partial\Omega_2$. Having obtained (5.11) and (5.16), by Theorem 2.1 we conclude that S has a fixed point y_2 such that

$$\min\{v, r\} \leqslant \|y_2\| \leqslant \max\{v, r\}. \tag{5.17}$$

We remark that this y_2 may *not* be different from y_1. □

Case (a): $r < \min\{w, v\}$. The relations (5.15) and (5.17), respectively, reduce to

$$r \leqslant \|y_1\| \leqslant w \quad \text{and} \quad r \leqslant \|y_2\| \leqslant v.$$

Coupling the above two inequalities, we see that y_1 may not be different from y_2, and we can only conclude that (1.7), (1.3), (1.4) has an eigenfunction y such that

$$r \leqslant \|y\| \leqslant \min\{w, v\}. \tag{5.18}$$

However, from the earlier part of the proof it is noted that (1.7), (1.3), (1.4) has twin eigenfunctions y_1 and y_2 such that the relation (5.2) holds. Using this fact together with (5.18), we immediately obtain

$$0 < \|y_1\| \leqslant r \leqslant \|y_2\| \leqslant \min\{w, v\}.$$

Case (b): $\min\{w, v\} < r < \max\{w, v\}$. There are two subcases to consider, namely, $w \geqslant v$ and $w \leqslant v$. Suppose that $w \geqslant v$. Then, $r \in (v, w)$ and from (5.15) and (5.17), respectively, we get

$$r \leqslant \|y_1\| \leqslant w \quad \text{and} \quad v \leqslant \|y_2\| \leqslant r.$$

A combination of the above two inequalities immediately gives

$$v \leqslant \|y_2\| \leqslant r \leqslant \|y_1\| \leqslant w. \tag{5.19}$$

Similarly, if $w \leqslant v$, then (5.15) and (5.17) lead to

$$w \leqslant \|y_1\| \leqslant r \leqslant \|y_2\| \leqslant v. \tag{5.20}$$

In view of (5.19) and (5.20), we conclude that (1.7), (1.3), (1.4) has twin eigenfunctions y_1 and y_2 such that

$$\min\{w, v\} \leqslant \|y_1\| \leqslant r \leqslant \|y_2\| \leqslant \max\{w, v\}.$$

Case (c): $r > \max\{w, v\}$. Here, from (5.15) and (5.17) we get

$$w \leqslant \|y_1\| \leqslant r \quad \text{and} \quad v \leqslant \|y_2\| \leqslant r.$$

So once again y_1 may not be different from y_2, and we can only conclude that (1.7), (1.3), (1.4) has an eigenfunction y such that

$$\max\{w, v\} \leqslant \|y\| \leqslant r. \tag{5.21}$$

As in Case (a), we combine (5.21) and the fact that (1.7), (1.3), (1.4) has twin eigenfunctions y_1 and y_2 satisfying (5.2), to obtain

$$\max\{w, v\} \leqslant \|y_1\| \leqslant r \leqslant \|y_2\|.$$

Remark 5.5. In [44], upper and lower bounds for twin eigenfunctions are also provided for two particular cases of (1.7), (1.3), (1.4).

Case 1: $\alpha = \gamma = 1$, $\beta = \delta = 0$. In this case, the upper and lower bounds obtained in [44] are the same as those in Theorem 5.2, though we have noted in Remark 5.2 that the condition (5.1) in present paper is an improvement.

Case 2: $\alpha = \delta = 1$, $\beta = \gamma = 0$. Here, $\rho = L = 1$ and $K_m = \min\{1, m, (1-m)/m\} = m$. Subsequently, in Theorem 5.4 we have

$$\phi(u) = \sup_{m \in (0,1/2)} (1-2m) m^{u+1} h^*(m).$$

On the other hand, in [44] the function corresponding to $\phi(u)$ is

$$\phi_1(u) = \tfrac{1}{2} \sup_{m \in (0,1)} (1-m^2) m^u h'(m),$$

where $h'(m) = \min_{t \in [m,1]} h(t)$. Denoting $w_1 = [\phi_1(a)]^{-1/(a-1)}$ and $v_1 = [\phi_1(b)]^{-1/(b-1)}$, we see that these values *cannot*, in general, be compared to w, v in Theorem 5.4. However, it has been noted in Remark 5.2 that the condition (5.1) in present paper is an improvement.

We can combine the upper and lower bounds obtained in [44] and in Theorem 5.4, to give a *sharper* result as follows.

Theorem 5.6. *We define*

$$M = \min\{\min\{w, v\}, \min\{w_1, v_1\}\}, \quad N = \min\{\max\{w, v\}, \max\{w_1, v_1\}\},$$
$$M' = \max\{\min\{w, v\}, \min\{w_1, v_1\}\} \quad \textit{and} \quad N' = \max\{\max\{w, v\}, \max\{w_1, v_1\}\}.$$

Let $r > 0$ be given. Suppose that (5.6) *holds. Then, the boundary value problem*

$$y'' + h(t)(y^a + y^b) = 0, \quad t \in (0,1),$$
$$y(0) = y'(1) = 0$$

has twin eigenfunctions y_1 *and* y_2 *such that*
(a) *if* $r<M$, $0<\|y_1\|\leqslant r\leqslant \|y_2\|\leqslant M$;
(b) *if* $M<r<M'$, $M\leqslant \|y_1\|\leqslant r\leqslant \|y_2\|\leqslant M'$;
(c) *if* $M'<r<N$, $M'\leqslant \|y_1\|\leqslant r\leqslant \|y_2\|\leqslant N$;
(d) *if* $N<r<N'$, $N\leqslant \|y_1\|\leqslant r\leqslant \|y_2\|\leqslant N'$;
(e) *if* $r>N'$, $N'\leqslant \|y_1\|\leqslant r\leqslant \|y_2\|$.

Example 5.7. Consider the boundary value problem

$$y''+\frac{6t}{\sqrt{3-t-t^3}+(3-t-t^3)^3}(y^{1/2}+y^3)=0, \quad t\in(0,1),$$
$$-y(0)-3y'(0)=0,\ 4y(1)+y'(1)=0.$$

It can be checked that (5.1) is fulfilled for any $0.170\leqslant r\leqslant 1.25$.
Here, $a=\frac{1}{2}$, $b=3$ and

$$h(t)=\frac{6t}{\sqrt{3-t-t^3}+(3-t-t^3)^3}$$

is found to be increasing on $[0,1]$. So $h^*(m)=h(m)$. Further, $\rho=7$, $L=\frac{3}{2}$ and also (2.3) provides $K_m=\frac{1}{5}(1+4m)$. Subsequently, we have

$$\phi(u)=\frac{3}{7}\left(\frac{2}{15}\right)^u \sup_{m\in(0,1/2)}(1+4m)^u(3-m)(1-2m)h(m).$$

By direct computation, we get

$$w=\left[\phi\left(\frac{1}{2}\right)\right]^2=4.74\times 10^{-4} \quad \text{and} \quad v=[\phi(3)]^{-1/2}=28.2.$$

Since $r\in(w,v)$, by Theorem 5.4(b) the boundary value problem has twin eigenfunctions y_1 and y_2 such that

$$4.74\times 10^{-4}\leqslant \|y_1\|\leqslant r\leqslant \|y_2\|\leqslant 28.2.$$

Noting the range of r, the above inequality leads to

$$4.74\times 10^{-4}\leqslant \|y_1\|\leqslant 0.170 \quad \text{and} \quad 1.25\leqslant \|y_2\|\leqslant 28.2. \tag{5.22}$$

Indeed, an eigenfunction is given by $y(t)=3-t-t^3$ and we note that $\|y\|=y(0)=3$ is within the range obtained in (5.22).

6. Twin eigenfunctions of (1.8), (1.3), (1.4)

Theorem 6.1. *Let* $r>0$ *be given. Suppose that*

$$L\int_0^1 G(s,s)h(s)\,\mathrm{d}s\leqslant re^{-\sigma r}. \tag{6.1}$$

Then, the boundary value problem (1.8), (1.3), (1.4) *has twin eigenfunctions* y_1 *and* y_2 *such that*

$$0<\|y_1\|\leqslant r\leqslant \|y_2\|. \tag{6.2}$$

Proof. Let $f(u)=\mathrm{e}^{\sigma u}$. Then, f fulfills (4.2) and (4.8). Further, we may take

$$q(t)=q_1(t)=h(t) \quad \text{and} \quad p(t)=p_1(t)=0.$$

It is clear that

$$f(u)\leqslant \mathrm{e}^{\sigma r}, \quad 0<u\leqslant r.$$

Therefore, (4.1) is satisfied provided that

$$\mathrm{e}^{\sigma r}\leqslant r\left\{L\int_0^1 G(s,s)[q_1(s)-p(s)]\,\mathrm{d}s\right\}^{-1}$$

which is the same as (6.1).

The conclusion is immediate from Theorem 4.3. □

Remark 6.2. Two particular cases of (1.8), (1.3), (1.4) have been discussed in [44].

Case 1: $\alpha=\gamma=1$, $\beta=\delta=0$. The condition corresponding to (6.1) is obtained [44] as

$$\int_0^1 (1-s)h(s)\,\mathrm{d}s\leqslant r\mathrm{e}^{-\sigma r}, \tag{6.3}$$

whereas in this case (6.1) is the same as

$$\int_0^1 s(1-s)h(s)\,\mathrm{d}s\leqslant r\mathrm{e}^{-\sigma r}. \tag{6.4}$$

Clearly, (6.4) is weaker than (6.3), and hence is an improvement over (6.3).

Case 2: $\alpha=\delta=1$, $\beta=\gamma=0$. In [44], the condition corresponding to (6.1) is

$$\int_0^1 h(s)\,\mathrm{d}s\leqslant r\mathrm{e}^{-\sigma r}. \tag{6.5}$$

On the other hand, (6.1) reduces to

$$\int_0^1 sh(s)\,\mathrm{d}s\leqslant r\mathrm{e}^{-\sigma r} \tag{6.6}$$

which is obviously weaker than (6.5), and therefore is an improvement over (6.5).

Example 6.3. Consider the boundary value problem

$$y''+h(t)\mathrm{e}^{4y}=0, \quad t\in(0,1),$$

$$y(0)=y(1)=0.$$

This problem arises in the diffusion of heat generated by positive temperature-dependent sources [1].

Let $r=1$ be given. Then, condition (6.1) reduces to

$$\int_0^1 s(1-s)h(s)\,\mathrm{d}s \leqslant \mathrm{e}^{-4}. \tag{6.7}$$

By Theorem 6.1, for any $h(t)$ that fulfills (6.7), the boundary value problem has twin eigenfunctions y_1 and y_2 such that

$$0<\|y_1\| \leqslant 1 \leqslant \|y_2\|.$$

Some examples of such $h(t)$ are $h(t)=\frac{1}{10}$, $\frac{1}{8}\sqrt{\sin \pi t}$, $\frac{1}{5}t$.

Once again, we shall establish upper and lower bounds for the twin eigenfunctions.

Theorem 6.4. *Let two different integers $j \geqslant 0$ ($j \neq 1$) and $i \geqslant 2$ be given. We define*

$$\psi(u)=\frac{1}{u!}\frac{2\delta+\gamma}{2\rho}\sup_{m\in(0,1/2)}\left(\frac{\sigma K_m}{L}\right)^u(\beta+\alpha m)(1-2m)h^*(m)$$

where $h^*(m)$ is given in (5.8),

$$w=[\psi(j)]^{-1/(j-1)} \quad \text{and} \quad v=[\psi(i)]^{-1/(i-1)}.$$

Let $r>0$ be given. Suppose that (6.1) *holds. Then, the boundary value problem* (1.8), (1.3), (1.4) *has twin eigenfunctions y_1 and y_2 such that conclusions* (a)–(c) *of Theorem* 5.4 *hold.*

Proof. Since (6.1) is fulfilled, by Theorem 6.1 the boundary value problem (1.8), (1.3), (1.4) has twin eigenfunctions y_1 and y_2 such that (6.2) holds.

To establish further upper and lower bounds for the twin eigenfunctions, let $m\in(0,\frac{1}{2})$ and C_m be a cone in B defined by (5.9). Further, we define the operator $S: C_m \to B$ by

$$Sy(t)=\int_0^1 G(t,s)h(s)\mathrm{e}^{\sigma y(s)}\,\mathrm{d}s, \quad t\in[0,1].$$

To obtain an eigenfunction of (1.8), (1.3), (1.4), we shall seek a fixed point of S in the cone C_m. As in the proof of Theorem 5.4, it can be verified that $S(C_m)\subseteq C_m$ and S is completely continuous.

Let $y\in C_m$ be such that $\|y\|=r$. Using Lemma 2.3 and (6.1), we get

$$Sy(t)\leqslant L\int_0^1 G(s,s)h(s)\mathrm{e}^{\sigma y(s)}\,\mathrm{d}s \leqslant L\int_0^1 G(s,s)h(s)\mathrm{e}^{\sigma r}\,\mathrm{d}s \leqslant r=\|y\|, \quad t\in[0,1].$$

Hence,

$$\|Sy\|\leqslant\|y\|. \tag{6.8}$$

If we set $\Omega=\{y\in B \mid \|y\|<r\}$, then (6.8) holds for $y\in C_m\cap\partial\Omega$.

Next, let $y \in C_m$. We find that

$$\begin{aligned}
\|Sy\| &\geqslant \int_0^1 G(m,s)h(s)\mathrm{e}^{\sigma y(s)}\,\mathrm{d}s \\
&\geqslant \int_m^{1-m} G(m,s)h^*(m)\mathrm{e}^{\sigma y(s)}\,\mathrm{d}s \\
&\geqslant \int_m^{1-m} G(m,s)h^*(m)\exp\left(\frac{\sigma K_m}{L}\|y\|\right)\mathrm{d}s \\
&\geqslant \int_m^{1-m} G(m,s)h^*(m)\left[\left(\frac{\sigma K_m}{L}\right)^j \frac{\|y\|^j}{j!} + \left(\frac{\sigma K_m}{L}\right)^i \frac{\|y\|^i}{i!}\right]\mathrm{d}s,
\end{aligned}$$

where in the last inequality we have used the relation

$$\mathrm{e}^u \geqslant \frac{u^j}{j!} + \frac{u^i}{i!}, \quad j \neq i,\ j \geqslant 0\ (j \neq 1), \quad i \geqslant 2.$$

On substituting the expression in (5.12), we simplify and then take supremum over m to get

$$\|Sy\| \geqslant \psi(j)\|y\|^j + \psi(i)\|y\|^i. \tag{6.9}$$

Following a similar technique as in Theorem 5.4, from (6.9) we obtain

$$\|Sy\| \geqslant \|y\| \tag{6.10}$$

for $y \in C_m \cap \partial\Omega'$ as well as for $y \in C_m \cap \partial\Omega''$, where

$$\Omega' = \{y \in B \mid \|y\| < w\} \quad \text{and} \quad \Omega'' = \{y \in B \mid \|y\| < v\}.$$

Now that we have obtained (6.8) and (6.10), by Theorem 2.1 S has a fixed point y_1 satisfying

$$\min\{w,r\} \leqslant \|y_1\| \leqslant \max\{w,r\}, \tag{6.11}$$

and also a fixed point y_2 (which may coincide with y_1) such that

$$\min\{v,r\} \leqslant \|y_2\| \leqslant \max\{v,r\}. \tag{6.12}$$

As in the proof of Theorem 5.4, a combination of (6.2), (6.11) and (6.12) yields (a)–(c) immediately. □

Remark 6.5. Upper and lower bounds for twin eigenfunctions are also offered for two particular cases of (1.8), (1.3), (1.4) in [44].

Case 1: $\alpha = \gamma = 1$, $\beta = \delta = 0$. It is noted that the upper and lower bounds obtained in [44] coincide with those in Theorem 6.4, though we have noted in Remark 6.2 that condition (6.1) in present paper is an improvement.

Case 2: $\alpha = \delta = 1$, $\beta = \gamma = 0$. Here, we have

$$\psi(u) = \frac{\sigma^u}{u!} \sup_{m \in (0,1/2)} (1 - 2m)m^{u+1}h^*(m).$$

However, in [44] the function corresponding to $\psi(u)$ is

$$\psi_1(u)=\frac{\sigma^u}{2(u!)}\sup_{m\in(0,1)}(1-m^2)m^u h'(m),$$

where $h'(m)=\min_{t\in[m,1]}h(t)$. Let $w_1=[\psi_1(j)]^{-1/(j-1)}$ and $v_1=[\psi_1(i)]^{-1/(i-1)}$. Clearly, these values *cannot*, in general, be compared to w,v in Theorem 6.4. Nevertheless, it has been noted in Remark 6.2 that condition (6.1) in present paper is an improvement.

Combining the upper and lower bounds obtained in [44] and in Theorem 6.4, we get a *sharper* result as follows.

Theorem 6.6. *Let M, N, M' and N' be defined as in Theorem 5.6 and let $r>0$ be given. Suppose that (6.6) holds. Then, the boundary value problem*

$$y''+h(t)e^{\sigma y}=0,\quad t\in(0,1),$$
$$y(0)=y'(1)=0$$

has twin eigenfunctions y_1 and y_2 such that conclusions (a)–(e) *of Theorem 5.6 hold.*

Example 6.7. Consider the boundary value problem

$$y''+\frac{2}{\exp(2t-2t^2+4)}e^{2y}=0,\quad t\in(0,1),$$
$$y(0)-2y'(0)=0,\qquad y(1)+2y'(1)=0.$$

By computation, condition (6.1) is satisfied provided that

$$0.0522\leqslant r\leqslant 1.83. \tag{6.13}$$

Here, $h(t)=2/\exp(2t-2t^2+4)$ and for any $m\in(0,\frac{1}{2})$,

$$h^*(m)=\min_{t\in[m,1-m]}h(t)=h\left(\frac{1}{2}\right)=2e^{-9/2}.$$

Further, $K_m=\frac{1}{3}(2+m)$, $L=1$ and $\rho=5$. Thus,

$$\psi(u)=e^{-9/2}\frac{1}{u!}\left(\frac{2}{3}\right)^u\sup_{m\in(0,1/2)}(2+m)^{u+1}(1-2m).$$

Let $j=0$ and $i=8$ be given. Then, we compute that $w=0.0222$ and $v=5.35$. Since $r\in(w,v)$, it follows from Theorem 6.4(b) that the boundary value problem has twin eigenfunctions y_1 and y_2 such that

$$0.0222\leqslant\|y_1\|\leqslant r\leqslant\|y_2\|\leqslant 5.35.$$

In view of (6.13), the above inequality provides

$$0.0222\leqslant\|y_1\|\leqslant 0.0522\quad\text{and}\quad 1.83\leqslant\|y_2\|\leqslant 5.35. \tag{6.14}$$

Indeed, an eigenfunction is given by $y(t)=t(1-t)+2$ and we note that $\|y\|=y(0.5)=2.25$ is within the range obtained in (6.14).

Example 6.8. Consider the boundary value problem

$$y''+ae^{\sigma y}=0, \quad t\in(0,1),$$
$$y(0)=y(1)=0$$

where $a,\sigma>0$. This problem has been well studied [1] and its solutions are

$$y_i(t)=-\frac{2}{\sigma}\left\{\log\left[\cosh\left(\frac{c_i}{2}\left(t-\frac{1}{2}\right)\right)\right]-\log\left(\cosh\frac{c_i}{4}\right)\right\}, \tag{6.15}$$

where c_i are solutions of the equation $c=\sqrt{2a\sigma}\cosh\frac{1}{4}c$.

Case 1: $a=1$, $\sigma=\frac{1}{2}$, $j=0$, $i=9$. It can be checked that condition (6.1) is satisfied provided that

$$0.183\leqslant r\leqslant 7.65. \tag{6.16}$$

Further, we find that $w=0.0625$ and $v=42.6$. Hence, it follows from Theorem 6.4(b) that the boundary value problem has twin eigenfunctions y_1 and y_2 such that

$$0.0625\leqslant\|y_1\|\leqslant r\leqslant\|y_2\|\leqslant 42.6.$$

In view of (6.16), we have

$$0.0625\leqslant\|y_1\|\leqslant 0.183 \quad\text{and}\quad 7.65\leqslant\|y_2\|\leqslant 42.6. \tag{6.17}$$

In fact, it is computed directly from (6.15) that $\|y_1\|=0.132$ and $\|y_2\|=10.3$.

In [44] the inequalities corresponding to (6.17) are found to be

$$0.0625\leqslant\|y_1\|\leqslant 0.715 \quad\text{and}\quad 4.31\leqslant\|y_2\|\leqslant 42.6.$$

Clearly, (6.17) gives sharper bounds. This is due to the improvement of condition (6.1).

Case 2: $a=7\cdot 10^{-4}$, $\sigma=3$, $j=0$, $i=16$. Again by computation, condition (6.1) is fulfilled if

$$1.17\cdot 10^{-4}\leqslant r\leqslant 3.42. \tag{6.18}$$

Further, we find that $w=4.38\cdot 10^{-5}$ and $v=11.5$. Hence, by Theorem 6.4(b) the boundary value problem has twin eigenfunctions y_1 and y_2 such that

$$4.38\cdot 10^{-5}\leqslant\|y_1\|\leqslant r\leqslant\|y_2\|\leqslant 11.5.$$

Once again in view of (6.18), it follows that

$$4.38\cdot 10^{-5}\leqslant\|y_1\|\leqslant 1.17\cdot 10^{-4} \quad\text{and}\quad 3.42\leqslant\|y_2\|\leqslant 11.5. \tag{6.19}$$

In fact, it is computed from (6.15) that $\|y_1\|=8.75\cdot 10^{-5}$ and $\|y_2\|=4.02$.

Corresponding to (6.19), in [44] we obtain

$$4.38\cdot 10^{-5}\leqslant\|y_1\|\leqslant 3.50\cdot 10^{-4} \quad\text{and}\quad 3.02\leqslant\|y_2\|\leqslant 11.5$$

which are not as sharp as (6.19). Again, this illustrates the improvement of condition (6.1).

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